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TITLE: **FUNDAMENTALS OF NUMERICAL MAGNETOHYDRODYNAMICS**

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Introduction

Magnetohydrodynamics is a fluid model for the motion of an ionized gas in a magnetic field. In it's ideal, non-dissipative form, the Lundquist equations, it has the same mathematical character as the model for gas dynamics. It gives, in the same way, a self-consistent description of the fluid dynamics, including the exchange of momentum and energy between the field and the fluid.

Mathematically, the Lundquist equations are a symmetric hyperbolic system of equations in conservation form. Theorems of existence and uniqueness of solutions for initial and boundary-value problems have been proven. Waves and shocks are among the solutions to the equations. In these respects, the Lundquist equations are completely similar to the gas dynamic equations.

However, because of the greater complexity of the physics which they describe, some aspects of the solutions are quite different. The magnetic field introduces a strong anisotropic character to the medium which causes wave propagation to depend on the direction of propagation with respect ot the magnetic field. In addition, there are several distinct speeds so that, in general, the responses to disturbances are quite complex.

To capture the principal features of the solutions in numerical calculations, several problems must be addressed. Some of these problems are unique to MHD: for example, preserving the solenoidality of the magnetic field. Others are similar to ordinary gas dynamics, such as energy conservation, numerical stability, and computational diffusion, but are more complex or have different consequences for MHD than for ordinary fluid flow.

These fundamental problems in the numerical solution of the MHD equations are discussed as four topics: the dispersion of the Lundquist equations and the dispersion and stability of finite difference approximations; the conservation laws of MHD and the achievement of conservation in the numerical solutions; a discussion of convective transport and its role in computational diffusion; and finally, a method for preserving the solenoidality of the magnetic field.

The Fundamental Equations of Plasma Physics

In plasma simulation, one solves the fundamental equations of plasma physics for the self-consistent evolution of the electric and magnetic fields, and for the individual particle orbits. A plasma, which is composed of electrons and ions, each with charge $\pm q$ and mass m_p, generates an electric and magnetic field, **E** and **B**. Individual particle orbits are determined by the dynamical equations,

$$\frac{d\mathbf{x}_p}{dt} = \mathbf{u}_p \tag{1}$$

and

$$\frac{d\mathbf{u}_p}{dt} = \frac{q_s}{m_p}\left(\mathbf{E} + \frac{\mathbf{u}_p \times \mathbf{B}}{c}\right). \tag{2}$$

The motion of individual particles causes fluctuations in the net charge,

$$\rho = \sum_{p,s} q_s \int d^3x' \; \delta(\mathbf{x}' - \mathbf{x}_p)\, S(\mathbf{x}' - \mathbf{x}), \tag{3}$$

and current,

$$\mathbf{J} = \sum_{p,s} q_s \int d^3x' \; \mathbf{u}_p\, \delta(\mathbf{x}' - \mathbf{x}_p)\, S(\mathbf{x}' - \mathbf{x}). \tag{4}$$

The shape function, S, which is familiar from plasma simulation, is a positive function with bounded support, h, and with unit normalization. In a physical plasma, h is negligible, but the number density of particles is so large that on any length scale of interest, ρ and **J** are smoothly varying. In a simulated plasma, this is not necessarily true because there are fewer particles. For mathematical convenience, the particle distributions are regularized in Eqs. (3) and (4) so that ρ and

J are continuous and differentiable even on small length scales compared with the particle separation.

The net charge and current produced by the particles in turn act on the electric and magnetic fields. The action is described by Maxwell's equations: the homogeneous equations are Faraday's laws,

$$\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 , \tag{5}$$

and the solenoidal condition on **B**,

$$\nabla \bullet \mathbf{B} = 0 . \tag{6}$$

These equations express intrinsic properties of the fields. The inhomogeneous equations which relate the sources to the fields are Poisson's equation,

$$\nabla \bullet \mathbf{E} = 4 \pi \rho , \tag{7}$$

and Ampere's law,

$$\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi \mathbf{J}}{c} . \tag{8}$$

The self-consistent solution of these equations is quite complex, requiring full knowledge of all the individual particle orbits and the resulting electric and magnetic fields. This can be done for Vlasov's equation for short times and small length scales by using the methods of plasma simulation.

The Derived Equations of Magnetohydrodynamics

One is fortunate that many physical problems can be modeled with less than complete knowledge of all the individual particle orbits. One successfully models fluid flows, for example, by solving the Navier-Stokes equations for the mass, momentum and energy of the flow. These equations average over the complicated motion of individual particles to reduce the number of independent variables needed to characterize the flow.

Similarly, magnetohydrodynamics (MHD) also averages over the motion of individual charged particles interacting with a magnetic field. Compared to plasma simulation, it describes the behavior of plasmas on long time and space scales, sufficiently long that rapid fluctuations in local charge density average to zero and that individual particles are deflected so much by short range interactions that they forget their initial conditions. The scattering, due to collisions, localizes the effect of individual particles to distances less than a mean free path, which is defined as the distance a particle can travel before being scattered through an angle equal to 90°. Usually, but not by necessity, the motion of particles is assumed to be non-relativistic.

The simplest model for magnetohydrodynamics is expressed by the Lundquist equations, which comprise Ampere's law,

$$\mathbf{J} = \frac{c}{4\pi}(\nabla \times \mathbf{B}) , \tag{9}$$

Faraday's law,

$$\frac{\partial \mathbf{B}}{\partial t} = - c \nabla \times \mathbf{E} \tag{10}$$

where **E** is given by Ohm's law,

$$\mathbf{E} + \frac{\mathbf{u} \times \mathbf{B}}{c} = 0. \tag{11}$$

u is the center of mass velocity of the fluid. The evolution of the sources of the magnetic field is described by a mass continuity equation,

$$\frac{\partial \rho}{\partial t} + \nabla \bullet \mathbf{u}\,\rho = 0, \qquad (12)$$

the momentum equation,

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \nabla \bullet \mathbf{u}\,\rho\,\mathbf{u} = -\nabla p + \mathbf{J} \times \mathbf{B}, \qquad (13)$$

and energy equation,

$$\rho \frac{\partial i}{\partial t} + \nabla \bullet \mathbf{u}\, i = -p\,(\nabla \bullet \mathbf{u}), \qquad (14)$$

where i is the internal energy per unit mass. These equations are derived from the particle equations in Ref. (1).

Magnetohydrodynamic Waves

Consider an infinite, homogeneous fluid in static equilibrium, with density ρ_0, pressure p_0, and magnetic field B_0. Small amplitude perturbations about this state evolve as described by the linearized equations of motion,

$$\frac{d\rho_1}{dt} + \rho_0(\nabla \bullet \mathbf{u}_1) = 0 \qquad (15)$$

$$\frac{dp_1}{dt} + \gamma p_0(\nabla \bullet \mathbf{u}_1) = 0 \qquad (16)$$

$$\rho_0 \frac{d\mathbf{u}_1}{dt} + \nabla (p_1 + \mathbf{B}_0 \mathbf{B}_1) - (\mathbf{B}_0 \bullet \nabla)\,\mathbf{B}_1 = 0 \qquad (17)$$

$$\frac{d\mathbf{B}_1}{dt} + \mathbf{B}_0(\nabla \bullet \mathbf{u}_1) - (\mathbf{B}_0 \bullet \nabla)\,\mathbf{u}_1 = 0 \qquad (18)$$

For simplicity, a system of units is chosen in which $c=4\pi$. If one substitutes a solution of the form,

$$\mathbf{B}_1 = \mathbf{B}_1(t=0)\, e^{i(\omega t - kx)} \qquad (19)$$

the partial differential equations above can be written as a system of algebraic equations,

$$-\omega \rho_1 + \rho_0 (\mathbf{k} \bullet \mathbf{u}_1) = 0 \qquad (15')$$

$$-\omega p_1 + \gamma p_0 (\mathbf{k} \bullet \mathbf{u}_1) = 0 \qquad (15')$$

$$-\omega \rho_0 \mathbf{u}_1 + k (p_1 + \mathbf{B}_0 \bullet \mathbf{B}_1) - (\mathbf{B}_0 \bullet \mathbf{k})\,\mathbf{B}_1 = 0 \qquad (17')$$

$$-\omega \mathbf{B}_1 + \mathbf{B}_0 (\mathbf{k} \bullet \mathbf{u}_1) - (\mathbf{B}_0 \bullet \mathbf{k})\,\mathbf{u}_1 = 0. \qquad (18')$$

These equations have a solution of the assumed form when the determinant of coefficients is zero,

$$0 = \pm \rho c^2 (\rho c^2 - B_n^2)\,[\rho c^4 - (\rho a^2 + B^2) c^2 + a^2 B_n^2] \qquad (20)$$

where $a^2 = \gamma p_0 / \rho_0$ is the ordinary sound speed, and B_n is the component of the magnetic field

in the direction of the wave travel, $B_n = \mathbf{B} \bullet k / |k|$.

The characteristic equation is eighth order, with four distinct roots. Each root corresponds to a possible independent mode of the system of linear equations. The roots are the particle characteristics,

$$c = 0, \tag{20a}$$

Alfven waves,

$$c = \pm \frac{B_n}{\rho}, \tag{20b}$$

which cannot travel perpendicular to B, and the slow and fast waves, which are roots of the term in brackets in Eq. 20,

$$c^2 = \frac{1}{2}\left(a^2 + A^2 \pm \sqrt{(a^2 + A^2) - 4a^2 A n^2}\right) \tag{20c}$$

where $A^2 = B^2/\rho$, and $A^2_n = B^2_n / A$. Parallel to the magnetic field, $A_n = A$, and the two solutions to Eq. (20c) are,

$$c = \pm a \tag{21a}$$

and

$$c = \pm A . \tag{21b}$$

Perpendicular to the magnetic field, the two roots are,

$$c = 0; \quad c = \pm (a^2 + A^2)^{1/2} . \tag{22}$$

The last roots are magnetoacoustic waves.

The wave speeds are represented in a Friedrich's diagram, which depicts the wave fronts that result from a point disturbance, such as would occur were a pebble dropped into a still pond of water. There are three wave fronts, two of which result from waves which propagate parallel to the magnetic field. The three waves are called fast, slow and Alfven waves. There are two cases, depending upon whether $a < A$ or $a > A$, as shown in Fig. (1).

For the linearized equations, the group and phase velocities are equal

$$\frac{d\omega}{dk} = \frac{\omega}{k}, \tag{23}$$

and thus the roots of the dispersion relation are the characteristic speeds at which information is propagated and the wave fronts define the domain of influence of each point in the medium. The wave fronts also define the domain of dependence of the solution at each instant. If one were to construct a diagram in one dimension of all the points from which a signal can propagate to a given point x_s in a time t, the diagram would look like Fig. (2). For numerical stability, the domain of dependence must always exceed the domain of influence.

A disturbance at $t = 0$ for any x such that $x_s - ct \le x \le x_s + ct$ where c is the maximum wave speed, will influence the solution at x_s at time t. When the system is finite, the boundaries also will begin to influence the solution after a sufficiently long time. If the signal transit time from boundary to boundary is short compared with the time scale of interest, the problem becomes more like a boundary value, or elliptic problem than an initial value, or hyperbolic problem.

Numerical Stability: Implicit and Explicit Schemes

If Eqs. (15-18) are differenced in time and Fourier transformed in space, they may be written,

$$\rho_1^1 - \rho_1^0 + \rho_0 \, i (k \bullet u_1^0) \Delta t = 0 \tag{15''}$$

$$p_1^1 - p_1^0 + \gamma p_0 \, i (k \bullet u_1^\theta) \, \Delta t = 0 \qquad (16'')$$

$$\rho_0 (u_1^1 - u_1^0) + i k (p_1^\varphi + B_0 B_1^\varphi) \Delta t - i (B_0 \bullet k) B_1^\varphi \Delta t = 0 \qquad (17'')$$

$$B_1^1 - B_1^0 + B_0 (i k \bullet u_1^\theta) \Delta t - i B_0 \bullet k \, u_1^\theta \Delta t = 0 , \qquad (18'')$$

where $u^\theta = \theta u' + (1 - \theta) u^0$, and $p \varphi = \varphi p^1 + (1 - \varphi) p^0$. The scheme is called a θ scheme. Various values of θ and φ correspond to well known algorithms. With $\theta = 1$ and $\varphi = 0$, one has the explicit leapfrog or Courant scheme; with $\theta = 1/2$ and $\varphi = 1/2$, the impiicit Richardson scheme; and with $\theta = 1$ and $\varphi = 1$, the implicit backward Euler scheme. To derive the dispersion relation for the discretized equations, one simply replaces the ordinary frequency ω by $\Omega(\omega)$ and the ordinary wave number k by $\kappa(k, \Omega)$. To obtain the numerical dispersion, one first solves for Ω and then for $\omega(k)$.

With the time variation, $e^{i \omega t}$, as before, the finite difference approximation to the time derivative becomes,

$$\frac{\rho_1^1 - \rho_1^0}{\Delta t} = i \omega \rho_1 e^{i \omega t} \frac{\sin \frac{\omega \Delta t}{2}}{\frac{\omega \Delta t}{2}} , \qquad (24)$$

where the reference time, t, is midway between ρ^1 and ρ^0. The intermediate value, u^θ, can be written,

$$u_1^\theta = 1/2 (u_1^1 + u_1^0) + (\theta - 1/2) - (u_1^1 - u_1^0) , \qquad (25)$$

$$= u_1 e^{i \omega t} = \left[\cos \frac{\omega \Delta t}{2} + 2 i (\theta - 1/2) \sin \frac{\omega \Delta t}{2} \right]$$

It is now convenient to define a new frequency Ω, and wave number κ,

$$\Omega = \omega \left[\frac{\tan \frac{\omega \Delta t}{2}}{\frac{\omega \Delta t}{2}} \right] , \qquad (26)$$

$$\kappa^\theta = k \left[1 + i (\theta - 1/2) \Omega \Delta t \right] ,$$

$$\kappa^\varphi = \left[1 + i (\varphi - 1/2) \Omega \Delta t \right] k .$$

The numerical phase velocity is simply

$$c'^2 = \frac{\Omega^2}{\kappa^\theta \kappa^\varphi}$$

and the dispersion relation is of exactly the same form as before with c' substituted for c. The roots of the numerical dispersion relation are calculated from the original roots,

$$c'^2 = c^2 \qquad (27)$$

or

$$\Omega^2 = c^2 \kappa^{\theta} \kappa^{\varphi} \tag{28}$$

requiring the solution of a transcendental equation.

For example, for waves propagating perpendicular to the magnetic field ($B_n = 0$), the wave speed is given by

$$c = \pm (a^2 + A^2)^{1/2} , \tag{29}$$

where A is the Alfven speed,

$$A = \frac{B}{\rho^{1/2}} , \tag{30}$$

and the wave is the magnetoacoustic wave. The frequency is given by,

$$\Omega^2 = k^2 (a^2 + A^2) (1 + i (\theta - 1/2) \Omega \Delta t) (1 + i (\varphi - 1/2) \Omega \Delta t). \tag{31}$$

When $\theta = \varphi = 1/2$ (the Richardson scheme), $-\pi \leq \omega \Delta t \leq \pi$ for all values of the right hand side. As Δt increases, ω approaches the Nyquist frequency, $\omega \rightarrow \pm \pi / \Delta t$. Because ω is real for all values of k and Δt, the Richardson scheme is unconditionally stable.

The Courant or leapfrog scheme corresponds to $\theta = 1, \varphi = 0$. One solves first the momentum equation u_1, and then Faraday's law, the continuity and pressure equations for B^1, ρ^1 and p^1 using u^1). The dispersion relation for the Courant scheme is given by

$$\Omega^2 = \frac{k^2 c^2}{1 - \left[\frac{k c \Delta t}{2}\right]^2} . \tag{32}$$

The scheme is conditionally stable. When $kc\Delta t > 2$, Ω is pure imaginary and thus ω is also pure imaginary, and exponentially growing modes occur. That is, when a signal propagates more than one wavelength in Δt, the scheme is unstable. The domain of influence exceeds the domain of dependence.

In general, when θ and φ are unequal to 1/2, the frequency is complex. For example, when $\theta = \varphi$, one can easily solve for the imaginary part of Ω,

$$Im(\Omega) = \frac{k^2 (a^2 + A^2) (\theta - 1/2) \frac{\Delta t}{2}}{1 = (\theta - 1/2)^2 k^2 (a^2 + A^2) (\frac{\Delta t}{2})^2} \tag{33}$$

When $\theta < 1/2$, Im $\Omega < 0$ (and thus Im (ω)) and the modes will grow exponentially. When $\theta > 1/2$, Im $\Omega > 0$, and the modes will decay exponentially. When Δt is very large,

$$Im (\Omega) \frac{\Delta t}{2} = \approx \frac{1}{(\theta - 1/2)} \tag{34}$$

or for $\theta = 1$

$$Im (\tan \frac{\omega \Delta t}{2}) \approx \frac{4}{\Delta t} \tag{35}$$

Thus, implicit time differencing will damp modes; explicit time differencing will be unstable unless $\theta + \varphi \geq 1$

The discretization in time will cause the phase velocity for short wavelength modes to be small. Physically, the ratio of ω/k should be constant for all k. Numerically, ω cannot be larger than the Nyquist frequency. Thus, as k increases, ω/k must decrease. Numerical dispersion will cause a train of short wavelength ripples to trail a disturbance.

Energy Conservation and Time Centering

Linear stability is a necessary, but not sufficient, condition for stability. The Lundquist equations are nonlinear, and one must have nonlinear stability as well. To achieve this, the equations must either be energy conservative or dissipative. If total energy increases with time, the equations are nonlinearly unstable.

In the analysis of the linearized Lundquist equations, a particularly simple case was examined above. Certain conclusions from that analysis can be applied to other discretizations that are not so simple. This allows the freedom to choose time differencing to conserve energy without violating linear stability constraints.

Consider the Lagrangian, semi-discrete approximation to the Lundquist equations. In these equations, the convective derivative and the time derivative are combined in the material derivative, and Faraday's law is written in it's equivalent Lagrangian form.

$$\rho^0 (i^1 - i^0) = -p^\varphi (\nabla \bullet \mathbf{u}^\theta)\, \Delta t \tag{36}$$

$$\mathbf{B}^1 - \mathbf{B}^0 = \left[-\mathbf{B}^1 (\nabla \bullet \mathbf{u}^\theta) + (\mathbf{B}^\varphi \bullet \nabla)\, \mathbf{u}^\theta \right] \Delta t \tag{37}$$

$$\rho^0 (\mathbf{u}^1 - \mathbf{u}^0) = \left[-\nabla \left(p^\varphi + \frac{\mathbf{B}^0 \bullet \mathbf{B}^1}{2} \right) + \nabla \bullet \mathbf{B}^\varphi \mathbf{B}^\varphi \right] \Delta t . \tag{38}$$

These are not the same choices of time levels as in Eqs. (16" - 18"), but they can be analyzed similarly and shown to be stable with $\theta, \varphi \geq 1/2$. From these equations, one can form the energy integral and demonstrate that energy is conserved.

Consider a small volume of fluid whose surfaces move with the fluid velocity, and within which the magnetic field, density, and pressure are essentially constant. Let $\varphi = 1/2$,

$$\mathbf{B}^\varphi = \varphi \mathbf{B}^1 + (1 - \varphi) \mathbf{B}^0 \tag{39}$$

The change in magnetic field energy is given by,

$$\Delta E_B = \frac{1}{2} \left[(B^1)^2 V^1 - (B^0)^2 V^0 \right] = \frac{1}{2} \left[(\mathbf{B}^1 + \mathbf{B}^0) \bullet (\mathbf{B}^1 - \mathbf{B}^0)\, V^0 + \frac{1}{2} (\mathbf{B}^1)^2 (V^1 - V^0) \right] \tag{40}$$

Substituting from Eq. (37) yields

$$\Delta E_B = \left[\{ -\frac{1}{2} \mathbf{B}^0 \bullet \mathbf{B}^1 \}\, \nabla \bullet \mathbf{u}^\theta + \mathbf{B}^\varphi \bullet (\mathbf{B}^\varphi \bullet \nabla)\, \mathbf{u}^\theta \right] V^0 \Delta t , \tag{41}$$

where the relation,

$$V^1 - V^0 = (\nabla \bullet \mathbf{u}^\theta)\, \Delta t\, V^0 \tag{42}$$

is assumed.

The change in kinetic energy is given by

$$\Delta E_K = \rho^0 \left[(\mathbf{u}^1)^2 - (\mathbf{u}^0)^2 \right] V^0 = \rho^0 \left[\mathbf{u}^\theta \bullet (\mathbf{u}^1 - \mathbf{u}^0) - (\theta - \frac{1}{2}) (\mathbf{u}^1 - \mathbf{u}^0)^2 \right] V^0 \tag{43}$$

Substituting from Eq. (38) yields,

$$\Delta E_K = \left[-\mathbf{u}^\theta \bullet \nabla \left(p^1 + \frac{\mathbf{B}^0 \bullet \mathbf{B}^1}{2}\right) + \mathbf{u}^\theta \bullet (\nabla \bullet \mathbf{B}^\varphi)\, \mathbf{B}^\varphi \right] V^0 \Delta t - (\theta - \frac{1}{2})\, (\mathbf{u}^1 - \mathbf{u}^0)^2\, \rho^0 V^0 \quad (44)$$

The change in internal energy is given by integrating Eq. (36). This change, added to Eqs. (42) and (44) yields the change in total energy,

$$\Delta E = \left[-\nabla \bullet \mathbf{u}^\theta \left(p^\varphi + \frac{\mathbf{B}^0 \bullet \mathbf{B}^1}{2}\right) + \nabla \bullet \mathbf{B}^\varphi\, \mathbf{B}^\varphi \bullet \mathbf{u}^\theta \right] V^0 \Delta t - (\theta - \frac{1}{2})\, (\mathbf{u}^1 - \mathbf{u}^0)^2\, \rho^0 V^0 \quad (45)$$

The last term is negative definite when $\theta > 1/2$, and the differencing is dissipative as required. Total energy is easily conserved, simply by adding the lost kinetic energy to the internal energy,

$$\rho^0 (i^1 - i^0) = - p^\varphi (\nabla \bullet \mathbf{u}^\theta) \Delta t + (\theta - 1/2) (\mathbf{u}^1 - \mathbf{u}^0)^2 \quad (46)$$

The Eulerian form of the equations can be analyzed similarly. Consider the explicit difference equations,

$$\rho^0 (\mathbf{u}^1 - \mathbf{u}^0) = \left[- \nabla \bullet \mathbf{u}^0\, \rho^0 \mathbf{u}^0 - \nabla p^0 + \nabla \times \mathbf{B}^0 \times \mathbf{B}^0 \right] \Delta t \quad (47)$$

$$\rho^0 (i^1 - i^0) = - p^0 (\nabla \bullet \mathbf{u}^1) \Delta t - \nabla \bullet \mathbf{u}^0\, \rho^0 i^0 \quad (48)$$

$$\mathbf{B}^1 - \mathbf{B}^0 = \left[\nabla \times \mathbf{u}^1 \times \mathbf{B}^0 \right] \Delta t \quad (49)$$

Following a similar path to the one above, one forms the energy integral for these equations,

$$\Delta E = \left[-\mathbf{u}^1 \bullet \nabla\, \mathbf{u}^0\, \rho^0\, \mathbf{u}^0 - \nabla \bullet \mathbf{u}^1 (p^0 + \rho^0 i^0) - \nabla \bullet ((\mathbf{u}^1 \times \mathbf{B}^0) \times \mathbf{B}^0) \right] \Delta t \quad (50)$$

$$- 1/2\, (\mathbf{u}^1 - \mathbf{u}^0)^2\, \rho^0 + 1/2\, (\mathbf{B}^1 - \mathbf{B}^0)^2$$

Two features of this equation should be noted. Because the equations are explicit, the magnetic field and velocity are at different time levels in the equations and energy is transferred from the flow to the magnetic field. If the scheme is to be dissipative overall, the sum of the two quadratic terms must be negative. This requires that the flow be supersonic,

$$u^2 > \frac{B^2}{\rho} = A^2.$$

Second, because the equations are Eulerian, a convective term for the kinetic energy appears which is not in conservation form. This is a common result, and forces the use of a conservation equation for total energy, rather than for its component parts individually.

From this single result, it is not correct to conclude that all explicit schemes will be nonlinearly unstable for low speed flow. However, it is often observed in numerical calculations that explicit methods are much noisier than implicit methods, and this may indicate an underlying instability. In any case, it indicates that schemes with acceptable properties in applications to fluid dynamics may not have the same properties when applied to MHD.

Spatial Differencing

To solve the Lundquist equations numerically, one must discretize in space as well as in time. The least complex case, for MHD flow on a Lagrangian grid in one dimension, will be discretized.

The data for a numerical calculation is typically specified by giving the values of the dependent variables at grid points, x_v. These points are caused to move with the fluid velocity,

$$u_v = \frac{dx_v}{dt}.$$

To define the values of the solution at intermediate points, the values at grid points are interpolated.

For example, the velocity is given by a similar convolution to that used earlier in Eqs. (3) and (4) for particles,

$$u(x) = \sum_v \int dx' \, \delta(x - x') \, S(x' - x) \, u_v ,$$

where S is the shape function defined by the recursion relation,

$$S^m = \int_{-\infty}^{\infty} dx' \, S^0(x - x') \, S^{m-1}(x') .$$

In this family of shape functions, there is included nearest grid point interpolation, or NGP,

$$S^0 = \frac{1}{\Delta x} \left\{ \begin{array}{ll} 1, & \frac{|x|}{\Delta x} \le 1/2 \\ 0, & \text{otherwise} \end{array} \right\},$$

and linear interpolation,

$$S^1 = \frac{1}{\Delta x} \left\{ \begin{array}{ll} 1 - \frac{|x|}{\Delta x}, & \frac{|x|}{\Delta x} \le 1 \\ 0, & \text{otherwise} \end{array} \right\}$$

Substituting these shape functions into the differential equations yields discrete equations directly.

Consider, first, the continuity equation, Eq. (12). To convert from the Eulerian to the Lagrangian form of this equation, one substitutes the definition of the Lagrangian, or material, time derivative,

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} .$$

In Lagrangian form, the continuity equation is written,

$$\frac{d\rho}{dt} + \rho \frac{\partial u}{\partial x} = 0 .$$

If one chooses to specify the density at the center of each cell, and to use NGP interpolation for the density and linear interpolation for the velocity,the discretized continuity equation that results is,

$$\sum_c \left[\frac{d\rho_c}{dt} + \rho_c \sum_c u_v \frac{\partial S^1(x - x_v)}{\partial x} \right] S^0(x - x_c) = 0 .$$

S is defined so that $dS/dt = 0$. Since S^0 and $\partial S^1/\partial x$ are constant within each cell, the continuity equation can be written,

$$\frac{d\rho_c}{dt} + \rho_c \frac{u_{v+1} - u_v}{x_{v+1} - x_v} = 0.$$

It is easy to show from the definition of cell volumes that mass is conserved.

It is useful to define geometric coefficients,

$$d_{vc} = \frac{1}{V_c}\begin{bmatrix} 1, & v = c+1 \\ -1, & v = c \\ 0, & \text{otherwise} \end{bmatrix} , \quad V_c = x_{v+1} - x_v .$$

Wherever the velocity is differentiated in the differential equation, the geometric coefficients will appear in the difference equations.

Using the geometric coefficiants, Faraday's law and the energy equation are written,

$$\frac{dB_{xc}}{dt} = 0 ,$$

$$\frac{dB_{yc}}{dt} + B_{yc}\sum_v d_{cv}\, u_v - B_{xc}\sum_v d_{cv}\, v_v = 0 ,$$

$$\frac{dB_{zc}}{dt} + B_{zc}\sum_v d_{cv}\, u_v - B_{xc}\sum_v d_{cv}\, w_v = 0 ,$$

$$\frac{dp_c}{dt} + \gamma p_c \sum_v d_{cv}\, u_v = 0 .$$

The momentum equation can be written using the coefficients, but the form should be chosen so that energy is conserved. First, let us define the energy by integrating the total energy over a cell,

$$E_c = \int_{x_v}^{x_{v+1}} \left\{ \rho\,(i + u^2/2) + B^2/2 \right\} dx$$

$$= \left[\rho_c \left(i_c + \frac{1}{6}\left(u_v^2 + u_v u_{v+1} + u_{v+1}^2 \right) \right) + \frac{B_c^2}{2} \right] V_c .$$

The kinetic energy can be expressed in terms of a mass matrix,

$$(E_K)_c = \sum_{v'}\sum_{v''} u_{v'}\, M^c_{v'v''}\, u_{v''} \quad , \quad v', v'' = 1,2 ,$$

where

$$M^c_{v'v'} = \frac{1}{3}\,\rho_c V_c \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} .$$

One can then postulate a momentum equation using the mass matrix and the geometric coefficients,

$$\sum_{v'} M^c_{vv'}\,\frac{du^\alpha_{v'}}{dt} = \sum_c d_{vc}\left(p_c + \frac{B_c^2}{2}\right) V_c - \sum_c B^x_c\, d_{vc} B^\alpha_c\, V_c \ , \quad \alpha = x,y,z$$

Because the same geometric coefficients are used for the momentum equation as for Faraday's law and the energy equation, energy conservation is easily demonstrated,

$$\sum_c \frac{dE_c}{dt} = \sum_{v,c} u_v d_{vc} \left(p_c + \frac{B_c^2}{2} \right) - \mathbf{u}_v \bullet B_c B_c^x d_{vc}$$

$$+ \sum_{c,v} - p_c d_{vc} u_v - \frac{B_c^2}{2} d_{vc} u_v + \mathbf{B}_c \bullet (d_{vc} B_c^x) \mathbf{u}_v \equiv 0$$

Unfortunately, the equations are difficult to solve because the mass matrix must be inverted to solve for u_v. This difficulty can be avoided by "lumping" the mass matrix,

$$M_c = \rho_c V_c .$$

With a lumped mass matrix, the momentum equation requires no matrix inversion,

$$\frac{1}{2} (M_c + M_{c-1}) \frac{du_v}{dt} = \sum_c d_{cv} p_c V_c .$$

The energy equation does not change because the "lumped" mass matrix is used. However, the substitution may introduce an inconsistency. For example, a linear pressure gradient will not yield a constant force on a grid with irregularly spaced points.

Generalizing the formulation to two and three dimensions requires redefining the geometric coefficients. Coefficients for two dimensional grids are calculated using bilinear interpolation, and for three using trilinear interpolation. A description of the method in three dimensions is given in Brackbill [Meth. Comput. Phys., 16, 1, 1976], and of the properties of the equations in Brackbbill [J. Comput. Phys. 65, 314, 1986].

Summary: Finite Difference Equations for MHD in One Dimension

Implicit finite difference equations for the MHD equations in one dimension on a Lagrangian grid are listed. The equations combine the differencing in time and space described in the previous sections.

In the order in which they are solved each time step, the equations are:

$$V_c = x_{v+1}^0 - x_v^0$$

$$d_{vc} = \frac{1}{V_c} \begin{bmatrix} -1, & v=c \\ 1, & v+1=c \\ 0, & \text{otherwise} \end{bmatrix}$$

$$\rho_c^1 - \rho_c^0 + \rho_c^1 \sum_v d_{vc} u_v^\theta \Delta t = 0$$

$$B_{xc}^1 - B_{xc}^0 = 0$$

$$B_{yc}^1 - B_{yc}^0 + B_{yc}^1 \sum_v d_{vc} u_v^\theta \Delta t - B_{xc}^\varphi \sum_v d_{vc} v_v^\theta \Delta t = 0$$

$$B_{zc}^1 - B_{zc}^0 + B_{zc}^1 \sum_v d_{vc} u_v^\theta \Delta t - B_{xc}^\varphi \sum_v d_{vc} w_v^\theta \Delta t = 0$$

$$\rho_c^1 - \rho_c^0 + \rho_c^1 \sum_v d_{vc}\, u_v^\theta\, \Delta t = 0$$

$$M_c = \rho_c\, V_c\,, \qquad M_v = \frac{1}{2}\,(M_c + M_{c-1})$$

$$M_v(u_v^1 - u_v^0) = \sum_c d_{cv} \left\{ p_c^\varphi + \frac{1}{2}(B_{yc}^0\, B_{yc}^1 + B_{zc}^0\, B_{zc}^1) \right\} V_c\, \Delta t$$

$$M_v\,(v_v^1 - v_v^0) = -\sum_c d_{vc}\, B_{xc}^\varphi\, B_{yc}^\varphi\, V_c\, \Delta t$$

$$M_v(w_v^1 - w_v^0) = -\sum_c d_{vc}\, B_{xc}^\varphi\, B_{zc}^\varphi\, V_c\, \Delta t$$

$$x_v^1 - x_v^0 = u_v^\theta\, \Delta t$$

For energy conservation, $\varphi = 1/2$.

Rezoning

A Lagrangian grid is not always the best one to use for a numerical calculation. For example, in a shock problem, the smallest cells develop behind the shock where the density is highest, rather than in the shock front where the gradients are largest. Yet, the accuracy of the numerical solution would be enhanced were the cells smallest in the shock front, in order to resolve the gradients.

To move the grid points where they are needed, or to add points, one must transfer the data from one set of grid points to another. Essentially, the data must be transferred as accurately as is practical. Inaccuracies in conserving mass, momentum, energy and magnetic flux, and the diffusion of gradients must be minimized.

A simple, yet satisfactory, transfer method for one-dimensional problems using interpolation may be constructed. Consider the density first. A functional, M, may be defined by,

$$M(x) = \int_0^x dx'\, \rho\,(x')$$

To recover the density, given M, one simply differentiates. The density is given by interpolation, but the integral of the density over the domain is independent of the order of interpolation because the interpolation function is normalized,

$$M = \sum_{c=1}^{N} \rho_c\, V_c \int_{x_v}^{x_{v+1}} dx'\, S^m\,(x' - x_c) = \sum_{c=1}^{N} \rho_c\, V_c$$

When the density is linearly interpolated, the functional is given by,

$$M_v = \sum_c \int_{x_{v-1}}^{x_v} dx'\, \rho_c\, S^1\,(x' - x_c) + M_{v-1}$$

Evaluating the integral yields,

$$\frac{1}{8}\left[\rho_{c-2} + 6\rho_{c-1} + \rho_c\right] = \frac{M_v - M_{v-1}}{x_v - x_{v-1}}$$

The density is given by a matrix equation, even when the grid does not move. If the matrix is lumped, as was the mass matrix in the Lagrangian equations above, computational diffusion is introduced. The replacement,

$$\rho_c' = \frac{1}{8}\left[\rho_{c-2} + 6\rho_{c-1} + \rho_c\right],$$

is equivalent to one diffusion step with diffusivity 1/8.

If the grid is replaced by a new one, the functional may be interpolated from one grid to the other without loss of mass. If x_v is replaced by x'_v, the new density on the new grid is calculated in two steps. First M on the new grid is calculated by interpolation,

$$M_v' = \sum_{v'} M_{v'} S^{[n]}(x_{v'} - x'_v).$$

The density is then calculated by differencing the functional, assuming that the density on the new grid is defined by the same interpolation as on the old,

$$\frac{1}{8}\left[\rho'_{c-2} + 6\rho'_{c-1} + \rho_c\right] = \frac{M_v' - M_{v-1}'}{x'_v - x'_{v-1}}$$

For consistency, the order of interpolation for the functional should be one greater than the order for the density. Since it is inconvenient to use higher order than quadratic because of the difficulties inspecifying boundary conditions, this limits the order of interpolation of the density to linear. One could avoid the matrix inversion by using NGP interpolation for the density. The equivalent order advection scheme is donor cell, which is known to be diffusive. Thus, one is restricted to linear interpolation.

Similar functionals may be defined for the momentum, magnetic flux and energy. They are,

$$U(x) = \int_0^x dx' \rho u; \quad V(x) = \int_0^x dx' \rho v; \quad W(x) = \int_0^x dx' \rho w$$

$$A_z = -\int_0^x dx' B_y; \quad A_y = \int_0^x dx' B_z$$

$$E = \int_0^x dx' \rho\left(i + \frac{u^2 + v^2 + w^2}{2}\right)$$

The momentum functional, for example, is calculated with NGP interpolation for the density and linear interpolation for the velocity, and is written,

$$\frac{\rho_{c-1}}{8}\left[u_{v-2} + 6u_{v-1} + u_v\right] = \frac{U_c - U_{c-1}}{x_c - x_{c-1}}$$

The other components of the momentum are calculated from similar equations.

The magnetic field is calculated similarly to the density. The energy, however, will be higher order in the kinetic energy than in the internal energy because of its quadratic dependence on the velocity.To deal with this difficulty, one should use the lumped mass matrix, defined above, with consequent dissipation of kinetic energy.

The method, EPIC, of which this is a simplified version, can be extended to two and three dimensional problems. EPIC is described by Eastwood, this meeting.

The Solenoidal Condition

The solenoidal condition on the magnetic field, Eq. (6), is an initial condition for the Lundquist equations. If one forms the divergence of Faraday's law, Eq. (10), one finds that if the field is solenoidal initially, it will remain solenoidal.

Unfortunately, this is not true for numerical calculations of MHD flow in two and three dimensions. This is one important aspect of numerical magnetohydrodynamics that is not the same in two and th e dimensions as it is in one. Either through errors in boundary conditions, or errors in the finite difference approximation to Faraday's law, the field may develop a non-solenoidal component in time. An error in solenoidality will cause the magnetic field to exert a non-physical force along field lines.

An easy correction to Faraday's law, which alters neither the stability nor the solubility of the Lundquist equations, is as follows: One adds a term to Faraday's law which causes the decay of div(**B**),

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} + d\,\nabla(\nabla \bullet \mathbf{B}).$$

The additional term is adapted from a suggestion made by Marder [J. Comput. Phys. 68,48,1987] to control charge imbalance in plasma simulation codes. It makes no difference to the physical solution, because div(**B**) is zero anyway, but it does cause the error to diffuse away from source regions in numerical calculations. Now the evolution equation for non-zero div(**B**) contains a self -correcting term,

$$\frac{\partial(\nabla \bullet \mathbf{B})}{\partial t} = \nabla \bullet d\,\nabla(\nabla \bullet \mathbf{B})$$

where d is the diffusivity. One can choose d arbitrarily, but it should be large enough to keep pace with the production of errors, and small enough so that it does not impose a limit on the time step when differenced explicitly.

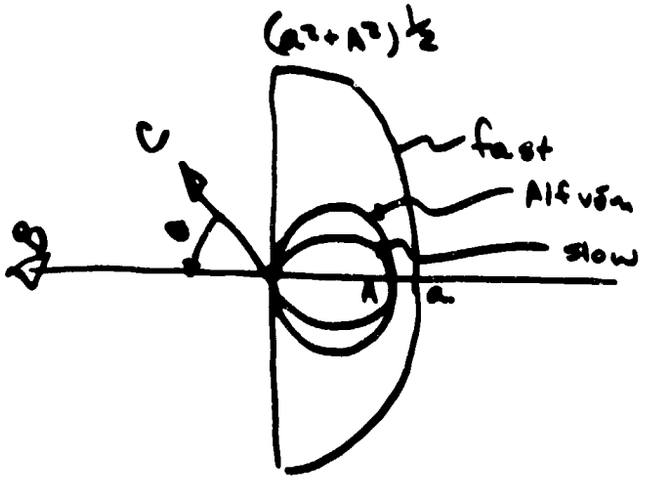
The correction term appears to be nonlinearly stable as well for it increases dissipation,

$$\mathbf{B} \bullet \frac{\partial \mathbf{B}}{\partial t} = -\,\mathbf{B} \bullet \nabla \times \mathbf{E} + d\,\mathbf{B} \bullet \nabla(\nabla \bullet \mathbf{B})$$

$$= -\,\nabla \bullet \mathbf{E} \times \mathbf{B} + \nabla \bullet d\,\mathbf{B}\,\nabla \bullet \mathbf{B} - \mathbf{E} \bullet \mathbf{J} - d(\nabla \bullet \mathbf{B})^2$$

The first terms are in conservation form. The third is the work done by the magnetic field on the fluid. The last is the dissipation in magnetic field energy due to the diffusion of the divergence of the field.

Case 1 $A < a$



Case 2 $A > a$

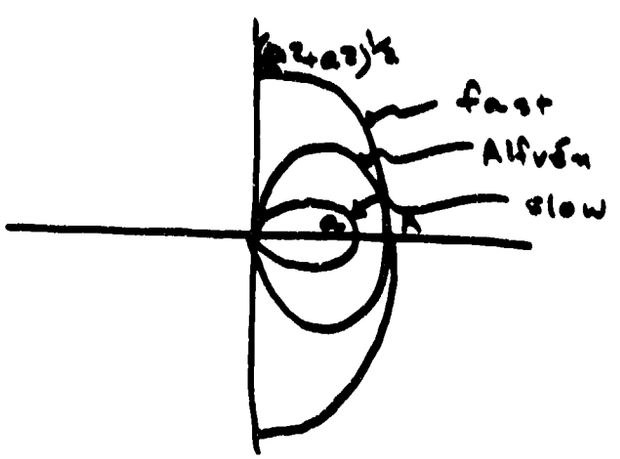
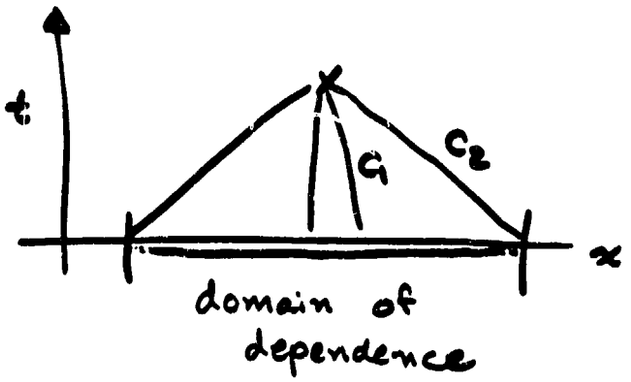
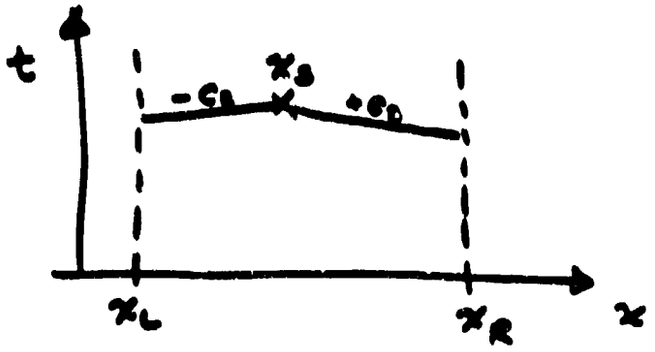


Figure 1. Interaction diagram

if there are many wave speeds, domain defined by largest speed



if system finite, after long enough time ($t >> L/c$), solution depends only on boundary conditions



solution depends on bdy data at $x_L(t-\varepsilon)$, $x_R(t-\varepsilon)$, ε small compared with time scale